Comparative dynamics in perfect foresight models

Meijdam, A.C.; Verhoeven, M.J.M.

Published in:
Computational Economics

Publication date:
1997

Computational Economics **12:** 115–124, 1998.

Comparative Dynamics in Perfect-Foresight Models

LEX MEIJDAM and MARIJN VERHOEVEN
CentER and Department of Economics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

(Received: 14 August 1997)

Abstract. This paper analyzes the technique of comparative dynamics (Judd, 1982) for the computation of the impact of perturbations on a steady state in a perfect-foresight model. The accuracy of this technique is demonstrated by numerical simulation experiments. Moreover, the technique is generalized to discrete-time models.

Key words: comparative dynamics, perfect foresight

1. Introduction

Dynamic perfect-foresight models form the core of modern macro-economic literature.[1] Due to technical difficulties, the analysis of this kind of models is often restricted to comparative statics and a graphical illustration of the saddlepath dynamics. A major drawback of this practice is that it is difficult to fully see through the consequences of perturbations of the steady state, especially when these perturbations are non-stationary. Therefore, sometimes the dynamics of the model are illustrated by numerical simulation experiments. Simulation, however, has the disadvantage that it is difficult to judge to what extent the outcomes depend on the specific parameter configuration used in the experiments. But there is a third way to analyze the dynamics of perfect-foresight models in continuous time. This is the method of comparative dynamics as presented by Judd (1982), which generates an analytically tractable approximation of the real solution. It is an elegant generalization of linear approximations as, for instance, employed in the method of comparative statics. In contrast to comparative statics, which is widely accepted as a basic tool for economists, the method of comparative dynamics seldom penetrates into the economic journals.[2] There may be three reasons for this. Firstly, many economists may feel ill at ease when highly non-linear dynamical relations are approximated by a simple linearization just as, admittedly, we did initially. Secondly, a large number of models are formulated in discrete time instead of continuous time so that Judd's method cannot always be readily applied. Finally, comparative dynamics may just have escaped the notice of a number of economists. The latter reason obviously is a poor excuse for not using comparative dynamics. The aim of this paper is to show that the former two are not a good excuse either.

2. The Method of Comparative Dynamics in Discrete Time

The basic perfect-foresight model of an economy in discrete time can be reduced to a system of difference equations:[3]

$$x_{t+1} = g^1(x_t, y_t, p_{s+t}(s = \ldots, -1, 0, 1, \ldots)), \tag{1a}$$

$$y_{t+1} = g^2(x_t, y_t, p_{s+t}(s = \ldots, -1, 0, 1, \ldots)), \tag{1b}$$

where x is a vector of non-predetermined variables (also known as jump or forward-looking variables), while y denotes the predetermined (or backward-looking) variables, and p is a vector of parameters or exogenous variables. The system is assumed to be initially in a stationary state $(\overline{x}, \overline{y})$ determined by $\overline{p}$, which is characterized by saddlepoint stability.[4] Now we are going to analyze the effect of a perturbation of the parameter vector. Let $p_t = \overline{p} + \gamma h_t$ $(t = 0, 1, \ldots)$. That is, the vector h_t denotes the time pattern of the perturbation and the scalar γ the magnitude, where h_t is assumed to be bounded and eventually constant. Linearization of system (1) around the initial stationary state by differentiation with respect to γ yields:

$$\begin{bmatrix} \dfrac{dx_{t+1}}{d\gamma} \\ \dfrac{dy_{t+1}}{d\gamma} \end{bmatrix} = J \begin{bmatrix} \dfrac{dx_t}{d\gamma} \\ \dfrac{dy_t}{d\gamma} \end{bmatrix} + \begin{bmatrix} \sum\limits_{s=-t}^{\infty} g_s^1 h_{s+t} \\ \sum\limits_{s=-t}^{\infty} g_s^2 h_{s+t} \end{bmatrix}, \tag{2}$$

where J is the Jacobian matrix and g_s^i $(i = 1, 2)$ denotes the derivative of g^i with respect to p_{s+t}, both evaluated in $(\overline{x}, \overline{y})$. If we define $X(\nu)$ and $Y(\nu)$ as the Z-transforms[5] of $dx/d\gamma$ and $dy/d\gamma$ respectively, and $G^i(\nu) \equiv \sum_{t=0}^{\infty} \sum_{s=-t}^{\infty} g_s^i h_{s+t} \nu^{-t-1}$ $(i = 1, 2)$ we can derive that:

$$\begin{bmatrix} \nu X(\nu) - \nu \dfrac{dx_0}{d\gamma} \\ \nu Y(\nu) - \nu \dfrac{dy_0}{d\gamma} \end{bmatrix} = J \begin{bmatrix} X(\nu) \\ Y(\nu) \end{bmatrix} + \begin{bmatrix} \nu G^1(\nu) \\ \nu G^2(\nu) \end{bmatrix}. \tag{3}$$

In order to determine $dx_0/d\gamma$, i.e., the jump in x at $t = 0$ induced by γ, a boundary value problem has to be solved. A set of initial conditions follows from the fact that y is predetermined: $dy_0/d\gamma = 0$. A set of final conditions results from the assumption that $dy_t/d\gamma$ and $dx_t/d\gamma$ are bounded, i.e., that the model follows the saddlepath to the steady state. Given these conditions, the problem can easily be solved (see Blanchard and Kahn (1980)). Here we only derive the solution for the case that the system consists of one predetermined and one jump variable. Then, solving (3) for the jump in the non-predetermined variable gives:

$$\frac{dx_0}{d\gamma} = \frac{j_{12}j_{21} - (j_{11} - \nu)(j_{22} - \nu)}{j_{21}\nu} Y(\nu) + \frac{j_{11} - \nu}{j_{21}} G^2(\nu) - G^1(\nu). \tag{4}$$

We now substitute the unstable root μ of the linearized system for ν. In that case $Y(\nu)$ is bounded and the first term on the RHS of Equation (4) is equal to zero which implies that:[6]

$$\frac{dx_0}{d\gamma} = \frac{(j_{11} - \mu)G^2(\mu)}{j_{21}} - G^1(\mu). \tag{5}$$

Given $dx_0/d\gamma$ (and $dy_0/d\gamma = 0$), the complete time path of $dx_t/d\gamma$ and $dy_t/d\gamma$ can be traced by applying system (2). Moreover, we can evaluate the induced change in dynamic evaluation functions in terms of x and y using Equation (3). Suppose e.g., that we are interested in:

$$u_0 = \sum_{t=0}^{\infty} w(x_t, y_t)(1+\theta)^{-t}. \tag{6}$$

Then the effect of γ on u_0 can be approximated by:

$$\frac{du_0}{d\gamma} = [\overline{w}_x, \overline{w}_y] \begin{bmatrix} X(1+\theta) \\ Y(1+\theta) \end{bmatrix}, \tag{7}$$

where $\overline{w}_x$ and $\overline{w}_y$ indicate the derivatives of w with respect to x and y evaluated in the stationary state. Using Equations (3) and (5) this can be rewritten as:

$$\frac{du_0}{d\gamma} = (1+\theta)[\overline{w}_x, \overline{w}_y][(1+\theta)I - J]^{-1} \begin{bmatrix} \frac{(j_{11} - \mu)G^2(\mu)}{j_{21}} - G^1(\mu) + G^1(1+\theta) \\ G^2(1+\theta) \end{bmatrix}. \tag{8}$$

Macroeconomists often use log-linearization instead of linearization.[7] One can derive approximations for the jump in the non-predetermined variables and in the dynamic evaluation function from a log-linear system analogous to the derivations above for a linear system. It can easily be shown that the resulting formula are exactly equivalent to Equations (5) and (8) however. So, whether a linearized or a log-linearized system is used does not affect the accuracy of the approximation by comparative dynamics.

3. The Accuracy of Comparative Dynamics

In this section the accuracy of the method of comparative dynamics will be illustrated by some examples. All examples are based on the standard Ramsey-model with a capital tax τ.[8] Production y is described by the following decreasing returns to scale technology:

$$y_t = f(k_t), \tag{9}$$

where k is the capital-labor ratio. Consumers maximize lifetime utility u, which equals the flow of CRRA-felicities associated with consumption c discounted at the rate of time preference θ:

$$u_0 = \sum_{t=0}^{\infty} \frac{c_t^{1-\beta}}{1-\beta}(1+\theta)^{-t}. \tag{10}$$

The first-order conditions then result in the following system of non-linear difference equations:

$$c_{t+1} = c_t \left(\frac{1 + (1-\tau) f'(k_{t+1})}{(1+n)(1+\theta)} \right)^{1/\beta}, \tag{11a}$$

$$k_{t+1} = \frac{k_t + f(k_t) - c_t}{1+n}, \tag{11b}$$

where n indicates the rate of population growth. Now we are going to analyze the effect of a once and for all change in the tax rate. That is, we assume that $\tau_t = \tau + \gamma$.[9] Linearization of system (11) around the stationary state $(\overline{c}, \overline{k})$ then gives:

$$\begin{bmatrix} \dfrac{dc_{t+1}}{d\tau} \\ \dfrac{dk_{t+1}}{d\tau} \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{(1-\tau) f''(\overline{k})\overline{c}}{\beta(1+\theta)(1+n^2)} & \dfrac{(1-\tau)(1+f'(\overline{k}))f''(\overline{k})\overline{c}}{\beta(1+\theta)(1+n)^2} \\ -\dfrac{1}{1+n} & \dfrac{1+f'(\overline{k})}{1+n} \end{bmatrix} \begin{bmatrix} \dfrac{dc_t}{d\tau} \\ \dfrac{dk_t}{d\tau} \end{bmatrix} + \begin{bmatrix} -\dfrac{f'(\overline{k})\overline{c}}{\beta(1+\theta)(1+n)} \\ 0 \end{bmatrix}. \tag{12}$$

Notice that this system consists of a jump variable $(dc_t/d\tau)$ and a predetermined variable $(dk_t/d\tau)$. Consequently, the system is saddlepoint stable if it has one root that lies outside the unit circle (μ). Dynamic efficiency $(f'(\overline{k}) > n)$ is a sufficient condition for this to be true. Following the method of comparative dynamics, the initial effect of a once-and-for-all change in the tax rate on consumption can be approximated by (cf. Equation (5)):

$$\frac{dc_0}{d\tau} = -G^1(\mu) = \frac{f'(\overline{k})\overline{c}}{\beta(1+\theta)(1+n)(\mu-1)}. \tag{13}$$

Moreover, the effect on lifetime utility can be approximated by (cf. Equation (8)):

$$\frac{du_0}{d\tau} = (1+\theta)[\overline{c}^{-\beta}, 0] \begin{bmatrix} \theta + \dfrac{(1-\tau)f''(\overline{k})\overline{c}}{\beta(1+\theta)(1+n)^2} & -\dfrac{(1-\tau)(1+f'(\overline{k}))f''(\overline{k})\overline{c}}{\beta(1+\theta)(1+n)^2} \\ \dfrac{1}{1+n} & 1+\theta-\dfrac{1+f'(\overline{k})}{1+n} \end{bmatrix}^{-1} \times \begin{bmatrix} \dfrac{f'(\overline{k})\overline{c}(\theta-\mu+1)}{\beta(1+\theta)(1+n)(\mu-1)\theta} \\ 0 \end{bmatrix} = \frac{\overline{c}^{1-\beta}\left(1+\theta-\dfrac{1+f'(\overline{k})}{1+n}\right) f'(\overline{k})(1+\theta-\mu)}{\left((1+\theta)\left(\theta+\dfrac{(1-\tau)f''(\overline{k})\overline{c}}{\beta(1+\theta)(1+n)^2}\right) - \dfrac{\theta(1+f'(\overline{k}))}{1+n}\right)\beta(1+n)(\mu-1)\theta}. \tag{14}$$

We evaluated the last two formulas for two levels of the tax rate ($\tau = 0$ and $\tau = 0.5$) and three rates of relative risk aversion ($\beta = 0.5$, $\beta = 1.5$ and $\beta = 10$). The results are presented in Table I. We also computed the change in initial consumption and lifetime consumption relative to the rate of change in the tax rate ($\Delta c_0/\Delta\tau$ and $\Delta u_0/\Delta\tau$) for these parameter sets numerically. This was done for three different magnitudes of the change in the tax rate. Table I also presents the relative error (ϵ) in the approximation by comparative dynamics. Of course, in general the accuracy of the approximation depends on the extent of the change in the tax rate: the relative error is proportional to the change in τ (compare the relative errors in the columns).[10]

Table I reveals that the accuracy of the approximation of the jump in initial consumption is almost independent of the initial level of the tax. This can be concluded from the relative errors ϵ_c. But the relative approximation errors ϵ_u seem to suggest that this is not true for the accuracy of the jump in utility. If the initial tax rate is positive comparative dynamics predicts the actual decrease in utility to a large extent, but in case of a zero initial tax rate the linear approximation predicts zero percent of the actual effect on u_0. The reason for this is that when $\tau = 0$ a marginal increase in the tax rate has no effect on utility. However, this also implies that, starting from $\tau = 0$, the actual effect of a discrete change in the tax rate is very small, going to zero if the tax change approaches zero. As a consequence, the approximation is as good as in any other point (see the absolute errors $\Delta u_0/\Delta\tau - du_0/d\tau$).

The accuracy of the approximation of the jump in c_0 increases when the felicity function is more concave (larger β). This is due to the fact that in that case the effect of a change in consumption on marginal utility and through that on the rate of change in c, is larger. Therefore, a smaller jump in initial consumption is required to bring the system on the saddlepath. In the experiments presented here this implies, in combination with the strict concavity of the felicity and production functions, that the error due to higher-order effects is also smaller. So increasing the non-linearity of a single function may in fact improve the linear approximation of the model variables. This is the case if the system equations, that determine the dynamics of the model, get more linear if the model equations are made less linear. Notice that the method of comparative dynamics underestimates the change in c for low values of β, but predicts too large a jump in case of $\beta = 10$. Furthermore, it is interesting to notice that the increased accuracy of the approximation of consumption is not reflected in the accuracy of the approximation of lifetime utility. The latter increases when β rises as the decrease in the prediction error in c is outweighed by the increased concavity of the felicity function.

In order to have a point of reference to judge the accuracy of comparative dynamics, we performed a comparative-statics analysis for the same parameter combinations. Comparative statics is a widely used linearization technique to calculate the steady-state or long-run effects of perturbations on model outcomes.[11] Just as with comparative dynamics, the outcomes of this analysis were compared to the numerical outcomes. The results of this experiment are presented in Table II. Matching the approximations in Table I and Table II learns that comparative dynam-

Table I. The accuracy of comparative dynamics.[a]

Parameter sets →			$\tau = 0$			$\tau = 0.5$		
			$\beta = 0.5$	$\beta = 1.5$	$\beta = 10$	$\beta = 0.5$	$\beta = 1.5$	$\beta = 10$
Comparative dynamics		$\frac{dc_0}{d\tau}$	2.0169	1.0459	0.2559	1.2100	0.5750	0.1155
Numerical simulations	$\Delta\tau = 0.1$	$\frac{\Delta c_0}{\Delta\tau}$	2.0795	1.0607	0.2562	1.2480	0.5853	0.1170
		ϵ_c	0.0301	0.0139	0.0014	0.0304	0.0176	0.0134
	$\Delta\tau = 0.01$	$\frac{\Delta c_0}{\Delta\tau}$	2.0231	1.0473	0.2559	1.2138	0.5760	0.1156
		ϵ_c	0.0031	0.0014	0.0001	0.0031	0.0017	0.0013
	$\Delta\tau = 0.001$	$\frac{\Delta c_0}{\Delta\tau}$	2.0175	1.0460	0.2559	1.2104	0.5751	0.1155
		ϵ_c	0.0003	0.0001	0.0000	0.0003	0.0002	0.0001
Comparative dynamics		$\frac{du_0}{d\tau}$	0	0	0	–8.4384	–5.8495	–1.2711
Numerical simulations	$\Delta\tau = 0.1$	$\frac{\Delta u_0}{\Delta\tau}$	–0.6192	–0.2546	–0.0006	–9.7773	–7.1032	–1.8084
		ϵ_u	1	1	1	0.1369	0.1765	0.2971
	$\Delta\tau = 0.01$	$\frac{\Delta u_0}{\Delta\tau}$	–0.0603	–0.0245	–0.0001	–8.5631	–5.9600	–1.3120
		ϵ_u	1	1	1	0.0146	0.0185	0.0312
	$\Delta\tau = 0.001$	$\frac{\Delta u_0}{\Delta\tau}$	–0.0060	–0.0024	0.0000	–8.4508	–5.8605	–1.2753
		ϵ_u	1	1	1	0.0015	0.0019	0.0033

[a] It is assumed that $n = 0.1$, $\theta = 0.1$ and $f(k_t) = k_t^{0.5}$. Furthermore, $\epsilon_c \equiv \frac{\Delta c_0/\Delta\tau - dc_0/d\tau}{\Delta c_0/\Delta\tau}$ and $\epsilon_u \equiv \frac{\Delta u_0/\Delta\tau - du_0/d\tau}{\Delta u_0/\Delta\tau}$.

ics performs significantly better than comparative statics in predicting the change in consumption, especially for high values of β. This suggests that the technique of comparative dynamics performs well according to standards generally accepted by economists. For the approximation of lifetime utility, the results of comparative dynamics are not always as good as those of comparative statics. This can be inferred from the case where the initial tax rate is set at $\tau = 0.5$ (the outcomes for the case of a zero initial tax rate cannot be compared in a meaningful way). For relatively low values of β comparative statics performs better than comparative dynamics, while comparative dynamics beats comparative statics for relatively high β's. In all cases, however, the inaccuracy of comparative dynamics is not very large in comparison with the standardly used method of comparative statics, justifying the use of comparative dynamics to analyze the short-run effects of parameter changes.

We have performed several other experiments. We have used different values for the capital share, for example. Barro and Sala-i-Martin (1995, Section 2.6.5) show that in a Ramsey model the change in the speed of convergence during the transition is more pronounced when capital share is smaller. This might be interpreted as an indication that the approximation by comparative dynamics is worse if the capital share is lower. It turns out that this it not true however: the accuracy of comparative dynamics slightly improves if the capital share is decreased. This is another example of the fact that increasing the non-linearity of a single function improves the linear approximation of the model variables. Judd (1987) applies the method of comparative dynamics to a model with elastic labor supply. In order to establish how well this method performs in this case, we have extended the model by including leisure in the felicity function in an additively separable way. This slightly improves the accuracy of comparative statics as well as comparative dynamics, but it does not change the main conclusion: comparative dynamics is as good as comparative statics. We have also performed experiments for the continuous-time version of the Ramsey model. The numerical outcomes of these exercises were virtually identical to the ones obtained for the discrete-time model. So also for continuous-time models the method of comparative dynamics generates precise enough results.

A major benefit of the method of comparative dynamics is that it makes the effects of non-stationary shocks tractable. Even if the time pattern of the shock is so complicated that it is impossible to draw a phase diagram to illustrate its effects, the method of comparative dynamics can be applied rather easily. The timing of the parameter change leaves all elements of the comparative-dynamics analysis unaffected, apart from the $G^i(\nu)$'s. Let us illustrate this with the Ramsey-model we used before. Assume for instance that $\tau_t = \tau + \gamma h_t$, where $h_t = \psi t \phi^{-t-1}$ with $\psi, \phi > 0$ is a hump-shaped function. Summing this function yields the Z-transform $H(\nu) = \sum_0^\infty \psi t(\phi+\nu)^{-t-1} = \psi(\phi+\nu)^{-2}$. That can be used to calculate $G^1(\nu) = -\psi(\phi+\nu)^{-2}\frac{\overline{c}}{\beta}f'(\overline{k})$ and $G^2(\nu) = 0$. Inserting this in Equations (5) and (7) together with the Jacobian of Equation (11) and the first vector of the RHS

Table II. The accuracy of comparative statics.[a]

Parameter sets →			$\tau = 0$			$\tau = 0.5$		
			$\beta = 0.5$	$\beta = 1.5$	$\beta = 10$	$\beta = 0.5$	$\beta = 1.5$	$\beta = 10$
Comparative statics		$\frac{d\overline{c}}{d\tau}$	–1.2472	–1.2472	–1.2472	–1.8141	–1.8141	–1.8141
	$\Delta\tau = 0.1$	$\frac{\Delta\overline{c}}{\Delta\tau}$	–1.3039	–1.3039	–1.3039	–1.8707	–1.8707	–1.8707
		ϵ_c	0.0435	0.0435	0.0435	0.0303	0.0303	0.0303
Numerical simulations	$\Delta\tau = 0.01$	$\frac{\Delta\overline{c}}{\Delta\tau}$	–1.2528	–1.2528	–1.2528	–1.8197	–1.8197	–1.8197
		$\overline{\epsilon}_c$	0.0045	0.0045	0.0045	0.0031	0.0031	0.0031
	$\Delta\tau = 0.001$	$\frac{\Delta\overline{c}}{\Delta\tau}$	–1.2477	–1.2477	–1.2477	–1.8146	–1.8146	–1.8146
		$\overline{\epsilon}_c$	0.0005	0.0005	0.0005	0.0003	0.0003	0.0003
Comparative statics		$\frac{d\overline{u}}{d\tau}$	–10.1857	–5.6149	–0.0355	–19.4853	–18.5795	–12.3969
	$\Delta\tau = 0.1$	$\frac{\Delta\overline{u}}{\Delta\tau}$	–10.8472	–6.2068	–0.0550	–21.0804	–22.1754	–38.7075
		$\overline{\epsilon}_u$	0.0610	0.0954	0.3533	0.0757	0.1622	0.6797
Numerical simulations	$\Delta\tau = 0.01$	$\frac{\Delta\overline{u}}{\Delta\tau}$	–10.2497	–5.6698	–0.0370	–19.6317	–18.8837	–13.5869
		$\overline{\epsilon}_u$	0.0062	0.0097	0.0386	0.0075	0.0161	0.0876
	$\Delta\tau = 0.001$	$\frac{\Delta\overline{u}}{\Delta\tau}$	–10.1921	–5.6203	–0.0357	–19.4998	–18.6095	–12.5087
		$\overline{\epsilon}_u$	0.0006	0.0010	0.0039	0.0007	0.0016	0.0089

[a] It is assumed that $n = 0.1$, $\theta = 0.1$ and $f(k_t) = k_t^{0.5}$. Furthermore, $\overline{\epsilon}_c \equiv \frac{\Delta\overline{c}/\Delta\tau - d\overline{c}/d\tau}{\Delta\overline{c}\Delta\tau}\Delta\tau$ and $\overline{\epsilon}_u \equiv \frac{\Delta\overline{u}/\Delta\tau - d\overline{u}/d\tau}{\Delta\overline{u}/\Delta\tau}$.

of Equation (13) gives the comparative-dynamics approximations of the jumps in consumption and utility. The accuracy of these outcomes is influenced by two factors: the total discounted value of the change in the tax rate, which of course negatively influences accuracy, and the time pattern of the shock. The simulation experiments we performed showed that if the tax rate change is more unevenly distributed, i.e., if the h_t-function gets more 'humped', the comparative-dynamics approximations of the jumps in consumption and utility become less precise. In other words, the increase in the linearization error emanating from the time intervals where the change in the tax rate is larger dominates the decrease in the linearization error caused by the smaller change in the tax rate in other time intervals. This is an intuitive result as the strictly concave nature of the felicity and production function suggests that higher-order effects increase more than proportionally with the size of the shock.

4. Conclusions

The method of comparative dynamics is a useful tool to analyze the effects of shocks in dynamic perfect foresight models in discrete and in continuous time. Numerical simulation experiments learn that, in the case of the Ramsey-model, it is as accurate as comparative statics which is a widely accepted technique. As a large part of modern macro-economic theory is closely related to the Ramsey-model, this result may be generalized to many of the dynamic perfect-foresight models economists use today. Therefore, comparative dynamics deserves a place in every economist's tool box. One should, however, always keep in mind that it is based on linear approximation, which implies that its accuracy tends to decrease if larger or more unevenly distributed parameter changes are studied, or if the non-linearity of the system is increased. On the other hand, it should be noticed that, as shown in simulation experiments, increasing non-linearity of a single function may in fact make the system more linear and thus raise the accuracy of the approximation.

Acknowledgements

We would like to thank Lans Bovenberg, Ben Heijdra and Arjan Lejour for comments on an earlier version. All remaining errors are ours.

Notes

[1] See e.g., Blanchard and Fischer (1989); Barro and Sala-i-Martin (1995).

[2] Exceptions are formed e.g., by Judd (1985), (1987) and Bovenberg (1989).

[3] Notice that we have extended the model used by Judd (1982) by allowing for a vector of parameters where the value of this vector at time s may influence the dynamics of the state variables at time t for $t \neq s$.

[4] Comparative dynamics can also be applied when the number of stable roots exceeds the number of predetermined variables. In that case it gives only one of the infinite number of possible solutions,

however. When the system has too many unstable roots there is no bounded solution and linearization makes no sense.

[5] The Z-transform is the discrete-time equivalent of the Laplace-transform used in Judd (1982). In general, if $f_t : \mathbb{R}^1 \to \mathbb{R}^n$ is a function of exponential order, the Z-transform of f_t is defined as $\sum_{t=0}^{\infty} f_t \nu^{-t} \equiv F(\nu) : \mathbb{R}^1 \to \mathbb{R}^n$. Notice that $\nu F(\nu) - \nu f_0 = \sum_{t=0}^{\infty} f_{t+1} \nu^{-t}$.

[6] If we compare this discrete-time result with the expression for the jump variable in continuous time as derived by Judd (1982) in his Equation (7), we find that if the dynamical system and Jacobian matrix are appropriately defined (compare Equations (4) of Judd (1982) and Equation (2) of this paper), the continuous time and discrete-time expressions for the non-predetermined variable only differ in $G^i(\nu)$. In discrete time $G^i(\nu)$ is defined as $\sum_{t=0}^{\infty} \sum_{s=-t}^{\infty} g_s^i h_{s+t} \nu^{-t-1}$, while in continuous time $G^i(\nu) = \int_{t=0}^{\infty} \int_{s=-t}^{\infty} g_s^i h_{s+t} e^{-\nu t} ds\, dt$ (see Judd (1982) p. 57). The definition of $G^i(\nu)$ also causes the only difference (a factor $1 + \theta$) between the expressions for the jump in the evaluation function in a continuous-time model (see Judd (1982), Equation (8)) and in a discrete-time model (see Equation (7) below).

[7] See e.g., Barro and Sala-i-Martin (1995), Chapter 2.

[8] See Blanchard and Fischer (1989), Chapter 2.

[9] Cf. the general equation $p_t = \overline{p} + \gamma h_t$ $(t = 0, 1, \ldots)$. Notice that h_t is constant in case of a once-and-for-all shock. Here we assumed $h_t = 1 \forall t \geq 0$ so that $d\tau \equiv d\gamma$.

[10] The reason for this is that for the relatively small values of the change in τ considered here the terms of third and higher order in the Taylor expansion are negligible, while the first-order term is included in the linear approximation. So, what is left is the second-order term, which turns up as a linear function of $\Delta\tau$ in the relative error.

[11] For a discussion of the method and accuracy of comparative statics, see Bovenberg and Keller (1984).

References

Barro, R.J. and Sala-i-Martin, X. (1995). *Economic Growth*. McGraw-Hill, New York.

Blanchard, O.J. and Fischer, S. (1989). *Lectures on Macroeconomics*. MIT Press, Cambridge, MA.

Blanchard, O.J. and Kahn, C.M. (1980). The solution of linear difference models under rational expectations. *Econometrica*, **48**, 1305–1311.

Bovenberg, A.L. (1989). The effects of capital income taxation on international competitiveness and trade flows. *American Economic Review*, **79**, 1045–1064.

Bovenberg, A.L. and Keller, W.J. (1984). Non-linearities in applied general equilibrium models. *Economics Letters*, **14**, 53–59.

Judd, K.L. (1982). An alternative to steady-state comparison in perfect foresight models. *Economics Letters*, **10**, 55–59.

Judd, K.L. (1985). Short-run analysis of fiscal policy in a simple perfect foresight model. *Journal of Political Economy*, **93**, 298–319.

Judd, K.L. (1987). The welfare cost of factor taxation in a perfect-foresight model. *Journal of Political Economy*, **95**, 675–709.